Pergamon

Int. J. Solids Structures Vol. 35, Nos. 5–6, pp. 543–555, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00034–6

COUPLED INSTABILITIES FOR A THICK ELASTIC PLATE UNDER THRUST

KONSTANTINOS A. LAZOPOULOS
Mechanics Department, School of Engineering Sciences, National Technical University of Athens, 5 Heroes of Polytechnion Ave. Gr-157 73, Zografou Campus, Athens, Greece

(*Received* 1 *March* 1996; *in revised form* 26 *November* 1996)

Abstract—For a thick incompressible hyperelastic plate under biaxial thrust, flexural and constitutive (homogeneous) instabilities are considered. In the present study their interaction (coupling) is discussed, when the critical conditions coincide for both kinds of instabilities. With the help of branching theory, especially the Fredholm Alternative theorem, the interactive post-critical equilibrium configurations are defined. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Stability interactive branching phenomena are quite common in structural engineering, especially in aerospace industry; it is well known that optimization of various structures leads, quite often, into interactive branching studies which are of paramount importance. Indeed, in the classical Koiter's thesis (1945) and various stability books, Thompson and Hunt (1973), many interactive (coupled) branching problems are discussed, concerning especially plate and shell structures. It is evident that the development of the mathematical branching procedures and especially in Banach spaces, see Vainberg and Trenogin (1974), substantially helped in understanding the various interactive branching phenomena; see papers of Bauer, Keller and Reiss (1975) and Shaeffer and Golubitsky (1981). Further, Thom's (1975) Catastrophe theory mathematically illuminates the imperfection sensitivity analysis, especially around umbilical branching points.

Although interactive branching phenomena are quite common in Finite Elasticity theory, their study is almost restricted into the location of critical conditions. No post-critical interactive (coupling) branching analysis exists, but the constitutive case of the interactive homogeneous deformations, Ball *et al.* (1983), Lazopoulos *et al.* (1995). The author, Lazopoulos (1995a,b), challenged by Sawyers and Rivlin's paper (1982) on the stability of a thick elastic plate under thrust, tried to transfer the branching techniques from the conventional elastic stability theory, to the Finite Elasticity theory, where non-uniqueness of the equilibrium solutions is a common phenomenon. In fact, the author (Lazopoulos (1995a,b)), reconsidered the branching of a thick incompressible elastic plate under thrust and described, using mathematical branching techniques, the post-critical flexure plate configuration. It is evident the problem exhibited one dimensional kernel (null space).

Insisting also on transferring interactive branching techniques into Finite Elasticity theory from the well known conventional elastic stability theory, which resorts almost exclusively strength of materials formulation, the present paper is presented just to become a prototype study for interactive branching analysis in Finite Elasticity. The present problem may be considered as the simplest coupled non-homogeneous branching problem in Finite Elasticity. Indeed, starting out from homogeneous deformations under biaxial thrust, a critical curve of the homogeneous stretches (strains) for the flexure post-critical response is defined. Furthermore, another critical curve for the stretches corresponding to constitutive (homogeneous) instabilities may be described. The common points of these critical curves define the isolated critical placements. Since the kernel (null space) of the coupled (interactive) problems is two dimensional, the post-critical equilibrium path has to be defined by invoking both branching theory in Banach spaces, Vainberg and Trenoging (1974) and Lazopoulos and Markatis (1996).

Stability problems with interactive (double) branching are quite common in the conventional elastic stability area. Indeed in the classical Koiter's thesis (1945) and the various representative books, Thompson and Hunt (1973), many coupled branching problems concerning especially plates and shells, are discussed. In addition, Vainberg and Trenogin's book (1974) deeply investigates the problem of the interactive branching for discrete and continuous systems, dealing with branching of the non-linear equations from the mathematical point of view. Further Thom's (1975) Catastrophe theory completely clarifies the mathematical behaviour of the imperfection sensitivity around the umbilical branching points. Signorini's perturbation method up to the second order of magnitude was applied around a critical finite elastic equilibrium placement. The first order equilibrium system defines the critical conditions along with the interactive modes. Nevertheless, the first order incremental displacement vector cannot be defined by the first order expansion of the equilibrium problem since it is an eigenvalue problem. Indeed, the first order displacement vector is described as a family of two parameter functions. The definition of the first order displacement vector could be achieved by the (Fredholm) existence of solution condition for the second order equilibrium system.

Since much of the theory of Signorini's expansions related to branching problems is exposed in Lazopoulos (1995a), that work will be an indispensable reference for the development of the present work. Further the format of Sawyers and Rivlin's (1974) paper concerning the bifurcation conditions for a thick elastic plate under uniaxial thrust will be followed. Therefore the easy access of that reference is also recommended.

It is noticed in Sections 2 and 3 a summary of the branching theory of the flexure buckling problem of a thick plate is outlined, as it has been described in Lazopoulos (1995a). Further, in Section 4 the constitutive instabilities are also briefly discussed, while in Section 5 the coupling (interaction) of the two instability modes (flexure and constitutive) is studied describing the post-critical equilibrium path. It should be pointed out that Section 5 constitutes the main contribution of the present work. In Section 6 an application is worked out implementing the theory.

2. THE EQUILIBRIUM PROBLEM OF A THICK INCOMPRESSIBLE PLATE

Let us consider an incompressible plate of dimensions $2l_1$, $2l_2$ and l_3 almost infinite. A co-ordinate system (X_1, X_2, X_3) is parallel to the edges of the plate. Furthermore the loading device is constituted by the thrust applied on the surfaces $X_1 = \pm l_1$ and the constant stress in the X_3 direction. The points $(X_1, X_2) = (\pm l, 0)$ are constrained to move on the line $(X_1, 0)$. When an experiment is performed, these supports undertake shear stress on the forced surfaces $X_1 = \pm l_1$, due to friction.

Just to be consistent, not only with the first order problem but also with the second order one, beam-like boundary conditions should be adapted, i.e., the resultant force per unit initial l_3-length applied on the surfaces $X_1 = \pm l_1$ should be equal to $2R_1 l_2$ with R_1 the first Piola-Kirchhoff stress in the X_1-direction. Lazopoulos (1995) has adapted that kind of boundary condition which has also been introduced by Rivlin (1949) for the study of the flexure problem.

Assuming that the thick elastic body is initially subjected to homogeneous deformations, every material point (X_1, X_2, X_3) is displaced to the finite placement (X_1^0, X_2^0, X_3^0) with

$$X_1^0 = \lambda_1 X_1 \tag{1a}$$

$$X_2^0 = \lambda_2 X_2 \tag{1b}$$

$$X_3^0 = \lambda_3 X_3. \tag{1c}$$

Since the elastic body is further deformed, every material point (X_1, X_2, X_3) is displaced to the current placement (x_1, x_2, x_3). Then the deformation gradient is defined by:

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial X_j}\right], \quad i,j = 1,2,3. \tag{2}$$

In addition the left Caushy-Green deformation tensor **B** is assigned by:

$$\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}} \tag{3}$$

where ($^{\mathrm{T}}$) denotes the transpose tensor; the principal strain invariants are defined by:

$$I_1 = B_{kk} = \operatorname{tr}\mathbf{B} \tag{4a}$$

$$I_2 = 0.5(B_{kk}B_{mm} - B_{km}B_{km}) = 0.5(\operatorname{tr} B)^2 - 0.5\operatorname{tr}(B^2). \tag{4b}$$

Yet the incompressibility condition is expressed by the constraint:

$$I_3 = \det(\mathbf{B}) = 1 \tag{5}$$

whereas the strain energy density is by the function:

$$W = W(I_1, I_2) \tag{6}$$

with $W(3,3) = 0$. Likewise, the first Piola-Kirchhoff stress tensor is defined by (Truesdell *et al.*, 1965; Wang *et al.*, 1973):

$$\mathbf{S} = -P\mathbf{F}^{-\mathrm{T}} + 2(W_1 + I_1W_2)\mathbf{B}\mathbf{F}^{-\mathrm{T}} - 2W_2(\mathbf{B}\cdot\mathbf{B})\mathbf{F}^{-\mathrm{T}} \tag{7}$$

where P is an arbitrary hydrostatic pressure imposed by the incompressibility constraint, eqn (5). The sub-indices of W indicate differentiation with respect to the principal invariants, i.e.,

$$W_i = \frac{\partial W(I_1, I_2)}{\partial I_i}, \quad W_{ij} = \frac{\partial W(I_1, I_2)}{\partial I_i \partial I_j}. \tag{8}$$

The equilibrium equations are defined by,

$$\frac{\partial S_{ij}}{\partial X_i} = 0 \quad \text{in } \Omega = ([-l_1, l_1]x[-l_2, l_2]xR) \tag{9}$$

and the beam like boundary conditions:

$$2l_2R_1 = \int_{-1/2}^{1/2} S_{11}\,\mathrm{d}X_2, \quad \int_{-1/2}^{1/2} S_{11}X_2\,\mathrm{d}X_2 = 0 \quad \text{on } X_1 = \pm l_1 \tag{10a}$$

$$S_{21} = S_{22} = 0 \quad \text{on } X_2 = \pm l_2 \tag{10b}$$

and $S_{3i} = \delta_{3i}R_3$ in Ω.

Let us consider a finite homogeneous deformation defined by the principal stretches ($\lambda_1, \lambda_2, \lambda_3$), Truesdell *et al.* (1965), Wang *et al.* (1973) engendered by the first Piola-Kirchhoff stress tensor,

$$S_{11} = R_1, \quad S_{22} = 0, \quad S_{33} = R_3 \quad \text{and} \quad S_{ij} = \delta_{ij}, \quad \text{for } i \neq j. \tag{11}$$

Perturbing the first Piola-Kirchhoff traction **S** so that

$$R_i = R_i^0 + \varepsilon^2 R_i^{**} + o(\varepsilon^2) \tag{12}$$

where $o(\varepsilon^3)$ denotes the terms of the same or higher order than ε^3, $(^0)$ the zeroth order terms, while $(^{**})$ the second order terms in ε, the resulting incremental displacement vector $\mathbf{u}$ is assumed to be expressed by the series expansion:

$$\mathbf{u} = \varepsilon\mathbf{u}^* + \varepsilon^2\mathbf{u}^{**} + \mathbf{O}(\varepsilon^3) \tag{13}$$

where the asterisks denote the order of the terms. It is noticed the expansion of R_i and the incremental displacement $\mathbf{u}$ are compatible with the branching theory, requiring for the latter to be the half order of the incremental traction. Hence the deformation gradient $\mathbf{F}$ equal

$$F_{ij} = \lambda_i\delta_{ij} + u_{i,j} \tag{14}$$

with $u_{i,j} = 0$ when i or j equals 3 but $u_{3,3} \neq 0$. Further the expansion of the hydrostatic pressure P is expressed by:

$$P = P^0 + \varepsilon P^* + \varepsilon^2 P^{**} + O(\varepsilon^3) \tag{15}$$

whereas the stress tensor $\mathbf{S}$ is given by:

$$\mathbf{S} = \mathbf{S}^0 + \varepsilon\mathbf{S}^* + \varepsilon^2\mathbf{S}^{**} + O(\varepsilon^3). \tag{16}$$

Furthermore, expanding the equilibrium eqns (9) into Taylor series in the parameter ε, Signorini's expansions around the finite equilibrium placement, Capriz and Guidugli (1979), defined by the $(\lambda_1, \lambda_2, \lambda_3)$ stretches, are formulated. Therefore, the equilibrium equations break into the following linearized systems, corresponding to equal order terms:
a) The first order of ε equilibrium system:

$$\mathrm{div}(\mathbf{S}^*) = \mathrm{div}(\boldsymbol{\Sigma}\mathbf{u}^*) = 0, \quad \boldsymbol{\Sigma} \text{ a linear operator.} \tag{17a}$$

$$\int_{-l_2}^{l_2} S_{11}^* \,\mathrm{d}X_2 = 0, \quad \int_{-l_2}^{l_2} S_{11}^* X_2 \,\mathrm{d}X_2 = 0 \quad \text{at } X_1 = \pm l_1 \tag{17b}$$

$$S_{22}^* = S_{21}^* = 0 \quad aX_2 = \pm l_2 \tag{17c}$$

$$u_2^*(\pm l_1, 0) = 0, \tag{17d}$$

$S_3^* = 0$ in Ω and
b) The second order of ε equilibrium system,

$$-\mathrm{div}(\Sigma\mathbf{u}^{**}) = B^{**}(\mathbf{u}^*, R_1^{**}, R_3^{**}) \quad \text{in } \Omega$$

$$R_1^{**} = 1/(2l_2)\int_{-l_2}^{l_2} S_{11}^{**} \,\mathrm{d}X_2, \quad \int_{-l_2}^{l_2} S_{11}^{**} X_2 \,\mathrm{d}X_2 = 0 \quad \text{at } X_1 = \pm l_1 \tag{18}$$

$$S_{22}^{**} = S_{21}^{**} = 0 \quad \text{at } X_2 \pm l_2 \tag{18c}$$

$$u_2^{**}(\pm l_1, 0) = 0, \tag{18d}$$

$$R_3^{**} = S_{33}^{**} = 0 \quad \text{in } \Omega \tag{18e}$$

where $\mathbf{B}^{**}(\mathbf{u}^*, \mathbf{R}_1^{**}, R_3^{**})$ is the non-linear part of the equation depending on the first order displacement field $\mathbf{u}^*$ and the loading parameters R_1^{**} and R_3^{**}.

Recalling the incompressibility condition requiring the third strain invariant to be equal to one, we get for the higher order terms the equations:

$$\frac{u^*_{1,1}}{\lambda_1}+\frac{u^*_{2,2}}{\lambda_2}+\frac{u^*_{3,3}}{\lambda_3}=0 \tag{19a}$$

$$\frac{u^{**}_{1,1}}{\lambda_1}+\frac{u^{**}_{2,2}}{\lambda_2}+\frac{u^{**}_{3,3}}{\lambda_3}+\frac{u^*_{1,1}u^*_{2,2}+u^*_{1,2}u^*_{2,1}}{\lambda_3}=0. \tag{19b}$$

The initial (pre-critical) homogeneous deformation with stretches $(\lambda_1, \lambda_2, \lambda_3)$ is described by the equations:

$$R^0_1 = S^0_{11}, \quad R^0_3 = S^0_{33} \tag{20a}$$

$$S^0_{22} = S^0_{21} = S^0_{31} = S^0_{23} = 0 \tag{20b}$$

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{20c}$$

The relation $S^0_{22} = 0$ defines P^0, the zeroth order term of P, by:

$$P^0 = 2\lambda_2^2(W_1+(I_1-\lambda_2^2)W_2). \tag{21}$$

The constant stretches $(\lambda_1, \lambda_2, \lambda_3)$ are defined by eqns (20) and (7).

$$R^0_1 = S^0_{11} = 2(\lambda_1^2-\lambda_2^2)(W_1+\lambda_3^2 W_2)/\lambda_1 \tag{22a}$$

$$R^0_3 = S^0_{33} = 2(\lambda_1^2-\lambda_2^2)(W_1+\lambda_1^2 W_2)/\lambda_3 \tag{22b}$$

$$R^0_2 = S^0_{22} = 0. \tag{22c}$$

For more information, see Sawyers *et al.* (1974).

3. THE FIRST ORDER FLEXURE EQUILIBRIUM PROBLEM

Following Sawyers *et al.* (1974), the first order equilibrium equations, eqn (9), are simplified in the compact form:

$$(1+\alpha_1)u^*_{1,11}+u^*_{1,22}-\lambda_2 p^*_{,1} = 0 \tag{23a}$$

$$u^*_{2,11}+(1+\alpha_2)u^*_{2,22}-\lambda_1 p^*_{,2} = 0 \tag{23b}$$

$$p^*_{,3} = 0 \tag{23c}$$

where

$$p^* = \frac{\lambda_2 \mathbf{P}^*}{2(W_1+\lambda_3^2 W_2)}$$

and α_1, α_2 are parameters defined by equations (3.10) of Sawyers *et al.* (1974) as:

$$\alpha_1 = \frac{2(\lambda_1^2-\lambda_2^2)}{W_1+\lambda_3^2 W_2}[W_{11}+(\lambda_1^2+2\lambda_3^2)W_{12}+\lambda_3^2(\lambda_3^2+\lambda_2^2)W_{22}] \tag{23d}$$

$$\alpha_2 = \frac{2(\lambda_2^2-\lambda_1^2)}{W_1+\lambda_3^2 W_2}[W_{11}+(\lambda_1^2+2\lambda_3^2)W_{12}+\lambda_3^2(\lambda_3^2+\lambda_1^2)W_{22}]. \tag{23e}$$

Further the incremental surface tractions vanish on the surfaces $X_2 = \pm l_2$. Hence

$$t_1^* = S_{21}^* = 2(W_1 + \lambda_3^2 W_2)(u_{1,2}^* + \lambda u_{2,1}^*) = 0 \quad X_2 = \pm l_2 \tag{24}$$

$$t_2^* = S_{22}^* = 2(W_1 + \lambda_3^2 W_2)\{(2+\alpha_2)u_{2,2}^* + \alpha_3 u_{3,3}^* - \lambda_1 p^*) = 0 \quad X_2 = \pm l_2 \tag{25}$$

where $\lambda = \lambda_2/\lambda_1$. Further, the boundary conditions on the surfaces $X_1 = \pm l_1$ are expressed by

$$\int_{-l_2}^{l_2} S_{11}^* \, dX_2 = 0, \quad \int_{-l_2}^{l_2} S_{11}^* X_2 \, dX_2 = 0 \quad \text{at } X_1 = \pm l_1 \tag{26a}$$

$$u_2^*(\pm l_1, 0) = 0, \tag{26b}$$

with

$$S_{11}^* = \sum_{c=1}^{3} \mathbf{K}_{11cc} u_{c,c}^* - \frac{p^*}{\lambda_1} \tag{27}$$

with K_{11cc} defined by the coefficients K_{AABB} in the Appendix; it should also be considered for the flexure mode,

$$u_{3,3}^* = 0. \tag{28}$$

In that case, see Lazopoulos (1995a), the solution for the first order homogeneous equilibrium system, eqn (23–27), is defined by:

$$u_1^* = -\sin(\omega X_1) U_1(X_2) \tag{29a}$$

$$u_2^* = \cos(\omega X_1) U_2(X_2) \tag{29b}$$

$$p^* = \cos(\omega X_1) \Gamma(X_2) \tag{29c}$$

where $\omega = \pi/2$ and

$$U_2(X_2) = M \cosh(\omega_1 X_2) + \cosh(\omega_2 X_2) \tag{29d}$$

and

$$U_1(X_2) = (\lambda\omega)^{-1}(\omega_1 M \sinh(\omega_1 X_2) + \omega_2 \sinh(\omega_2 X_2)) \tag{29e}$$

$$\Gamma(X_2) = (\lambda_2 \lambda \omega)^{-1}\{\omega_1(\omega_1^2 - (1+\alpha_1)\omega^2) M \sinh(\omega_1 X_2) + \omega_2(\omega_2^2 - (1+\alpha_1)\omega^2)\sinh(\omega_2 X_2)\} \tag{29f}$$

with

$$\omega_1^2, \omega_1^2 = 0.5\omega^2(1 + \lambda^2 + A(1-\lambda)^2 \pm \{1 + \lambda^2 A(1-\lambda)^2 - 4\lambda^2\}^{1/2}) \tag{29g}$$

where $A = 2(\lambda_1 + \lambda_2)^2 (W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22})/(W_1 + \lambda_3^2 W_2)$ and

$$M = -\frac{\omega_2^2 + \lambda^2\omega^2}{\omega_1^2 + \lambda^2\omega^2} \frac{\cosh(\omega_2 l_2)}{\cosh(\omega_1 l_2)}. \tag{30}$$

Likewise, the eigenvalue condition for the flexure mode in terms of the parameter λ is expressed by,

$$\frac{\tanh(\omega_2 l_2)}{\tanh(\omega_1 l_2)} = \frac{\omega_1}{\omega_2}\left[\frac{\omega_2^2 + \lambda^2\omega^2}{\omega_1^2 + \lambda^2\omega^2}\right]^2. \tag{31}$$

4. THE CONSTITUTIVE INSTABILITIES

Constitutive instabilities, especially in incompressible materials, have been initiated by Rivlin (1974), discussed by Hill (1982); further literature could be found in the review paper of Beatty (1987). With the term constitutive instabilities we denoted the instabilities of homogeneous deformations with post-critical equilibrium configurations again homogeneous. In the present case of isotropic incompressible materials, the energy per unit volume is expressed by:

$$V = W - R_1\lambda_1 - R_3\lambda_3 \tag{32}$$

with R_1 and R_3 the first Piola-Kirchhoff principal stresses, since $R_2 = 0$. Further the incompressibility condition satisfies the equation:

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{33}$$

Substituting for λ_3 from eqn (33) into eqn (32), it appears:

$$V = W - R_1\lambda_1 - R_3/(\lambda_1\lambda_2). \tag{34}$$

Following Lazopoulos *et al.* (1995) the equilibrium equations are defined by:

$$\frac{\partial V}{\partial \lambda_1} = \frac{\partial W}{\partial \lambda_1} - R_1 + \frac{R_3}{\lambda_1^2\lambda_2} = 0 \tag{35a}$$

$$\frac{\partial V}{\partial \lambda_2} = \frac{\partial W}{\partial \lambda_2} + \frac{R_3}{\lambda_2^2\lambda_1} = 0. \tag{35b}$$

Let us consider a homogeneous equilibrium Λ^0 placement defined by the stretches $(\lambda_1^0, \lambda_2^0, \lambda_3^0)$ under the action of the $\mathbf{R}^0 = (R_1^0, R_2^0, 0)$ vector of the first Piola-Kirchhoff stress vector. Perturbing the stress vector so that

$$R_1 = R_1^0 + \mathrm{d}R_1 \tag{36a}$$

$$R_2 = R_2^0 + \mathrm{d}R_2 \tag{36b}$$

the new (current) homogeneous equilibrium position is defined by the stretches:

$$\lambda_1 = \lambda_1^0 + \mathrm{d}\lambda_1 \tag{37a}$$

$$\lambda_2 = \lambda_2^0 + \mathrm{d}\lambda_2 \tag{37b}$$

$$\lambda_3 = \lambda_3^0 + \mathrm{d}\lambda_3. \tag{37c}$$

Considering, further, the incompressibility condition, eqn (33), $\mathrm{d}\lambda_3$ is linearly dependent on $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$. Expanding, likewise, the equilibrium equations, eqns (35a,b), in Taylor series around the equilibrium placement $(\Lambda^0, \mathbf{R}^0)$ we get:

$$\frac{\partial V(\Lambda, \mathbf{R})}{\partial \lambda_1} = \frac{\partial V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1} + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1^2}\mathrm{d}\lambda_1 + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1 \partial \lambda_2}\mathrm{d}\lambda_2 + \Phi_1(\mathrm{d}R_1, \mathrm{d}R_2, \mathrm{d}\lambda_1, \mathrm{d}\lambda_2) = 0 \tag{38a}$$

$$\frac{\partial V(\Lambda, \mathbf{R})}{\partial \lambda_2} = \frac{\partial V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_2} + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_2^2}\mathrm{d}\lambda_2 + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1 \partial \lambda_2}\mathrm{d}\lambda_1 + \Phi_2(\mathrm{d}R_1, \mathrm{d}R_2, \mathrm{d}\lambda_1, \mathrm{d}\lambda_2) = 0. \tag{38b}$$

The equilibrium system, eqns (38a,b), is written as,

$$\frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1^2} d\lambda_1 + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1 \partial \lambda_2} d\lambda_2 = -\Phi_1(dR_1, dR_2, d\lambda_1, d\lambda_2) \tag{39a}$$

$$\frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_1 \partial \lambda_2} d\lambda_1 + \frac{\partial^2 V(\Lambda^0, \mathbf{R}^0)}{\partial \lambda_2^2} d\lambda_2 = -\Phi_2(dR_1, dR_2, d\lambda_1, d\lambda_2). \tag{39b}$$

A nonunique solution of the system (39a,b) exists when the branching condition expressed by:

$$\frac{\partial^2 V}{\partial \lambda_1^2}\frac{\partial^2 V}{\partial \lambda_2^2} - \left[\frac{\partial^2 V}{\partial \lambda_1 \partial \lambda_2}\right]^2 = 0 \tag{40}$$

is valid with the branching (solution) vector

$$\gamma = \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ d\lambda \end{bmatrix} \tag{41}$$

where

$$d\lambda = \frac{\partial^2 V}{\partial \lambda_1^2} \Bigg/ \frac{\partial^2 V}{\partial \lambda_1 \partial \lambda_2}.$$

Likewise, $d\lambda_3$ is expressed as a linear function of $d\lambda_1$, $d\lambda_2$ with the help of the incompressibility condition (33) as:

$$d\lambda_3 = -(\lambda_3/\lambda_1 + (\lambda_3/\lambda_2)\, d\lambda)\, d\lambda_1.$$

The critical branching equilibrium placement is defined by the equilibrium equations (39a and b) and the branching condition (40).

5. THE INTERACTIVE BRANCHING

It is evident that eqns (31) and (29g) yield the critical curve between λ_1 and λ_2 so that for each pair of the stretches (λ_1, λ_2) lying on that critical curve, flexure instabilities arise. Furthermore, eqn (40) yields the eigenvalue condition for the appearance of constitutive instabilities. It is clear this is also a (critical) curve between the two first principal stretches (λ_1, λ_2). Therefore at the common points of the two critical curves eqns (31) and (40), we expect coupled interactive branching of the flexure and constitutive modes. The main purpose of the present work is the description of the post-critical equilibrium path. When the critical conditions, eqns (31) and (40), coincide, the first order equilibrium system, eqns (23) and (27) accept a double solution, the flexure solution,

$$\varphi^f = \xi_1[u_1^*, u_2^*, 0] \tag{42}$$

defined by eqns (29) and the homogeneous solution

$$\varphi^h = \xi_2[1, d\lambda, (-\lambda_3/\lambda_1 - (\lambda_3/\lambda_2)\, d\lambda)]. \tag{43}$$

According to branching theory for continuous systems (in Banach spaces), see Ch. 7 of Vainberg *et al.* (1974), the first order branching solution $\mathbf{v}$ is given by the linear combination of both solutions φ^f and φ^h. Hence

$$\mathbf{v} = \xi_1[u_1^*, u_2^*, 0] + \xi_2[1, d\lambda, (-\lambda_3/\lambda_1 - (\lambda_3/\lambda_2)\,d\lambda)] + \mathbf{o}(\xi_1^2 + \xi_2^2) \tag{44}$$

with ξ_1 and ξ_2 arbitrary.

The second order equilibrium system is given by eqns (18) and the corresponding incompressibility condition (19b). In fact the governing equilibrium equations for the second order displacement field are expressed by:

$$(1+\alpha_1)u_{1,11}^{**} + u_{1,22}^{**} - \lambda_2 p_{,1}^{**} = B_1(\xi_1\varphi^f, \xi_2\varphi^h) \tag{45a}$$

$$u_{2,11}^{**} + (1+\alpha_2)u_{2,22}^{**} - \lambda_1 p_{,2}^{**} = B_2(\xi_1\varphi^f, \xi_2\varphi^h) \tag{45b}$$

$$p_{,3}^{**} = B_3(\xi_1\varphi^f, \xi_2\varphi^h) \tag{45c}$$

$$\frac{u_{1,1}^{**}}{\lambda_1} + \frac{u_{2,2}^{**}}{\lambda_2} + \frac{u_{3,3}^{**}}{\lambda_3} = B_4(\xi_1\varphi^f, \xi_2\varphi^h) \tag{45d}$$

where a_1 and a_2 have already been defined and $p^{**} = 0.5P^{**}(W_1 + \lambda_3^2 W_2)^{-1}$. Furthermore the second order boundary conditions, eqns (18b–18e) have to be satisfied. That system (eqns 45, 18b–18e) has the form:

$$\mathbf{L}(\mathbf{u}^{**}) = \mathbf{B}(\xi_1\varphi^f, \xi_2\varphi^h, R_1^{**}, R_3^{**}) \tag{46}$$

where $\mathbf{L}$ stands for the linear part of the systems and B is the non-linear operator which depends upon the first order displacement vectors $\xi_1\varphi^f$ and $\xi_2\varphi^h$ and the second order forcing system R_1^{**}, R_3^{**}. It is evident that the linear homogeneous system,

$$\mathbf{L}(v) = 0 \tag{47}$$

is similar to the first order equilibrium system (23–27). Therefore the existence of double Kernel in the first order system signifies the second order solution for the non-linear problem, eqn (46) of the form:

$$\mathbf{u}^{**} = \xi_1\varphi^f + \xi_2\varphi^h + \mathbf{u}^{**p}. \tag{48}$$

Nevertheless, the existence of a solution to non-linear eqn (46) presupposes the Fredholm alternative condition, see Ch. 7 of Vainberg *et al.* (1974) and Lazopoulos (1995), which in our case is expressed by the algebraic equations:

$$\iint_\Omega (B_1\varphi_1^f + B_2\varphi_2^f + B_3\varphi_3^f)\,d\Omega + 2l_2 T_1^{**}\varphi_1^f + 4l_1 l_2 T_3^{**}\varphi_3^f = 0 \tag{49}$$

and

$$\iint_\Omega (B_1\varphi_1^h + B_2\varphi_2^h + B_3)\,d\Omega + 2l_2 T_1^{**}\varphi_l^h + 4l_1 l_2 T_3^{**}\varphi_3^h = 0 \tag{50}$$

where

$$T_1^{**} = R_1^{**} - \frac{1}{2l_2}\int_{-l_2}^{l_2}\left(\mathbf{S}_{11}^{**} - \sum_{c=1}^{3} K_{11cc}\mathbf{u}_{c,c}^{**} - \frac{p^{**}}{\lambda_1}\right) dX_2 \tag{51}$$

$$T_3^{**} = R_3^{**} - \frac{1}{4l_1 l_2}\int_{-l_2}^{l_2}\int_{-l_1}^{l_1}\left(\mathbf{S}_{33}^{**} - K_{3333}\mathbf{u}_{3,3}^{**} - \frac{p^{**}}{\lambda_3}\right) dX_1\, dX_2 \tag{52}$$

where, R_1^{**} and R_3^{**} are the incremental traction stresses as it is indicated by eqn (12).

Recalling the symmetries of the flexure and homogeneous solutions of the first order problem, eqns (49, 50) become two branching algebraic equations of the form:

$$\xi_1 \xi_2 = 0 \tag{53}$$

$$\xi_1^2 + \theta \xi_2^2 = \delta_1 R_1^{**} + \delta_2 R_3^{**} \tag{54}$$

where, θ, δ_1, δ_2 are known coefficients, defined by performing the algebra of the Fredholm alternative conditions, corresponding to φ^f and φ^h, eqns (49 and 50). Since the algebra is long, "Mathematica" computerized algebra pack will be an indispensable tool for the present study. It should be noticed that eqn (53) is the result of the non-interaction of an antisymmetric (flexure) and the symetric (constitutive) solutions. When the R.H.S. of eqn (54) is different from zero, the solutions of the branching (53, 54) system are:

$$\xi_1 = \pm(\delta_1 R_1^{**} + \delta_2 R_2^{**})^{1/2} \tag{55a}$$

$$\xi_2 = 0 \tag{55b}$$

or

$$\xi_1 = 0 \tag{56a}$$

$$\xi_2 = \pm[(\delta_1 R_1^{**} + \delta_2 R_2^{**})/\theta]^{1/2}. \tag{56b}$$

The solutions (55 or 56) for ξ_1 and ξ_2 of the system (53, 54) with respect to the incremental forcing system (R_1^{**}, R_3^{**}) define the first order post-critical displacements of the equilibrium system. They prove that the interaction of both modes is restricted only to the location of the critical state. Due to the symmetry of the modes, they do not interact in the post-critical region as the eqns (55 and 56) indicate.

6. APPLICATION

Let us consider a thick plate of an extended Mooney-Rivlin material with non-dimensionalized strain energy density function:

$$W = 1.28\,(I_1 - 3) - (I_2 - 3). \tag{57}$$

The plate is of unit length $l_1 = 1$, while l_2 is taken equal to 0.4, while l_3 is infinite. It is evident that the coefficients $a = 1.28$ and $b = -1$ have been specified just for the rise of coupled instabilities. Indeed, for this kind of material and the specific geometry of the plate, both eigenvalue (critical) eqns (31 and 40) are satisfied. Looking for flexure solutions of the mode $\omega = \pi/2$, the critical stretch ratio $\lambda_{cr} = \lambda_{2,cr}/\lambda_{1,cr}$ is defined by the eigenvalue equation:

$$\frac{\tanh(\omega_2 l_2)}{\tanh(\omega_1 l_2)} = \frac{\omega_1}{\omega_2}\left[\frac{\omega_2^2 + \lambda^2\omega^2}{\omega_1^2 + \lambda^2\omega^2}\right]^2 \tag{58a}$$

with ω_1, ω_2 defined by the eqns (29g). Since ω_1, ω_2 and λ are functions only of λ_1, λ_2, eqn (58a) is an algebraic equation of λ_1, λ_2. The other critical condition, eqn (40), for the constitutive instabilities is further satisfied at λ_{cr} when:

$$\frac{\partial^2 \bar{W}}{\partial \lambda_1^2}\frac{\partial^2 \bar{W}}{\partial \lambda_2^2} - \left[\left(\frac{\partial^2 \bar{W}}{\partial \lambda_1 \partial \lambda_2}\right)\right]^2 = 0 \tag{58b}$$

where $\bar{W}$ is equal to W but λ_2 has been substituted by $(\lambda_1\lambda_3)^{(-1)}$ from the incompressibility condition. Therefore eqn (58b) is also an algebraic equation for the stretches λ_1 and λ_2.

Equation (58a) is a critical curve with respect to λ_1, λ_2 stretches corresponding to flexure mode instability. Likewise, eqn (58b) is the other critical curve with respect to λ_1, λ_2 stretches corresponding to constitutive instabilities. The common solutions of the eqns (58, 59) correspond to coupled (interactive) instabilities. The closest common point of the curves (58a, 58b) to the initial stretches (of the unstressed body, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$) are defined numerically by:

$$\lambda_{1,cr} = 0.90 \tag{59a}$$

$$\lambda_{2,cr} = 1.20 \tag{59b}$$

$$\lambda_{3,cr} = 0.925 \tag{59c}$$

and

$$\lambda_{cr} = 1.333 \tag{59d}$$

with

$$R_1^0 = -0.5915 \quad \text{and} \quad R_3^0 = -0.5917. \tag{59e}$$

Further, eqn (29g) yields:

$$\omega_1 = \pi/2 \quad \text{and} \quad \omega_2 = \lambda_{cr}\pi/2 = 1.33\pi/2 \tag{60}$$

and the flexure mode $\varphi^f = (\varphi_1^f, \varphi_2^f, 0)$ is defined by eqns (29a and b) with:

$$\varphi_1^f = -\sin(\pi X_1/2)(-1.46\cosh(\pi X_2/2) + \cosh(1.33\pi X_2/2)) \tag{61a}$$

$$\varphi_2^f = 0.24\pi\cos(\pi X_1/2)(-1.46\cosh(\pi X_2/2) + 1.33\cosh(1.33\pi X_2/2). \tag{61b}$$

The corresponding constitutive instability exhibits a branching direction φ^h with

$$(\varphi_1^h, \varphi_2^h, \varphi_3^h) = (1, -0.064, -0.979). \tag{62}$$

Now the post-critical solution $\mathbf{v}$ is given by eqn (44) as a linear combination of both eigensolutions φ^f, and φ^h. The coefficients ξ_1 and ξ_2 will be defined by the Fredholm Alternative conditions, eqns (49 and 50). As it has been discussed, the solvability condition (49) yields the eqn (53), whereas the solvability Fredholm condition (50) is reduced, after performing the long and tedious algebra with the Mathematica computerized pack, to the equation:

$$\xi_1^2 = -1.5661 R_1^{**} - 3.1322 R_3^{**}. \tag{64}$$

Clearly, eqn (64) corresponds to eqn (54) with $\theta = 0$, $\delta_1 = -1.5661$, $\delta_2 = -3.1322$. It is obvious that eqn (53) is compatible with eqn (64) only when

$$\xi_2 = 0. \tag{65}$$

Recalling that the R.H.S. of eqn (64) should be positive, it always yields real solutions ξ_1 for compressive incremental stresses R_1^{**}, R_3^{**}. It is also pointed out that the critical stresses R_1^0 and R_3^0 are compressive too. Therefore, with compressive incremental traction the two incremental displacement equilibrium solutions $\mathbf{v} = (v_1, v_2, 0)$ are expressed by the functions:

$$v_1 = \pm[-1.5661 R_1^{**} - 3.1322 R_3^{**}]^{1/2}[-\sin(\pi X_1/2)(-1.46\cosh(\pi X_2/2) + \cosh(1.33\pi X_2/2))] \tag{65a}$$

$$v_2 = \pm[-1.5661R_1^{**}-3.1322R_3^{**}]^{1/2}[0.24\pi\cos(\pi X_1/2)(-1.46\cosh(\pi X_2/2)+1.33\cosh(1.33\pi X_2/2)).] \quad (65b)$$

$$v_3 = 0. \quad (65c)$$

Consequently, the first order term, i.e., the tangent direction, of the post-critical equilibrium path has been defined.

It is quite clear that in the present problem the constitutive mode does not appear in the post-critical region; hence only the flexure mode interacts.

7. CONCLUSION

The coupling (interaction) of the flexure and constitutive instabilities was studied for an incompressible thick hyperelastic plate under biaxial thrust. The post-critical equilibrium path was defined, by using the Fredholm solvability conditions. It was proved that the interaction of the flexure and the constitutive modes are restricted only to the location of the critical conditions (state). No-interaction of the modes takes place in the post-critical region. The present problem could be used as a compass for solving various coupled instability problems in Finite elasticity theory.

Acknowledgement—The author is indebted to the reviewers for their valuable comments and their help in bringing the paper into the present shape.

REFERENCES

Ball, J. M. (1977) Convexity conditions and existence theorems in nonlinear elasticity. *Archives of Rational Mechanics Analysis* **63,** 337–403.

Ball, J. M. and Schaeffer, D. G. (1983) Bifurcation and stability of homogeneous equilibrium configurations of an elastic body under dead load tractions. *Mathematical Proceedings of the Cambridge Philosophical Society* **94,** 315–339.

Bauer, L., Keller, H. B. and Reiss, E. L. (1975) Multiple eigenvalues lead to secondary bifurcation. *Siam Review* **17,** 101–122.

Beatty, M. F. (1987) Topics in finite elasticity: hyperelasticity of rubber, elastomers and biological tissues—with examples. *Applied Mechanics Review* **12,** 1699–1734.

Green, A. N. and Adkins, J. E. (1960) *Large Elastic Deformations.* Oxford University Press, London and New York.

Hill, R. (1982) Constitutive branching in elastic materials. *Mathematical Proceedings of the Cambridge Philosophical Society* **92,** 167–181.

Koiter, W. T. (1945) On the stability of equilibrium. Dissertation, Delft, Holland.

Lazopoulos, K. A. (1995a) Second order displacement fields under thrust. The incompressible case. *International Journal of Solids and Structures* **33**(3), 433–449.

Lazopoulos, K. A. (1995b) Branching of a thick elastic plate under uniaxial thrust. In *4th National Congress in Mechanics*, HSTAM, Xanthi, Greece.

Lazopoulos, K. A. and Markatis, S. (1995) Constitutive branching in incompressible elastic materials. In *Proceedings of 4th National Congress in Mechanics*, HSTAM, Xanthi, Greece.

Rivlin, R. S. (1949) Large elastic deformations of isotropic materials, part V, the problem of flexure. *Proceedings of the Royal Society* **A195,** 463–473.

Rivlin, R. S. (1974) Stability of pure homogeneous deformations of an elastic cube under dead loading. *Quarterly Applied Mathematics* **32,** 265–271.

Sawyers, K. N. and Rivlin, R. S. (1974) Bifurcation conditions for a thick elastic plate under thrust. *International Journal of Solids and Structures* **10,** 483–501.

Sawyers, K. N. and Rivlin, R. S. (1982) Stability of a thick elastic plate under thrust. *Journal of Elasticity* **12,** 101–125.

Schaeffer, D. G. and Golubitsky, M. A. (1981) Bifurcation analysis near a double eigenvalue of a model chemical reaction. *Archive for Rational Mechanics and Analysis* **75,** 315–347.

Thom, R. (1975) *Structural Stability and Morphogenesis.* Benjamin, New Jersey.

Thompson, J. M. T. and Hunt, G. W. (1973) *A General Theory of Elastic Stability.* Wiley, London.

Truesdell, C. and Noll, W. (1965) The non-linear field theories of Mechanics. *Handbuch Der Physik III/3*, ed. S. Fluegge. Springer Verlag, Berlin.

Vainberg, M. M. and Trenogin, V. A. (1974) *Theory of Branching of Solutions of Non-linear Equations.* Leyden, Noordhoff.

Wang, C. C. and Truesdell, C. (1973) *Introduction to Rational Elasticity.* Leyden, Noordhoff.

APPENDIX

The various K_{ABCD} coefficients shown in eqn (27) and eqns (51, 52) as well having been defined by Sawyers and Rivlin (1974) as follows:

$$K_{AABB} = 2[W_1 + (I_1 - \lambda^2) W_2]\delta_{AB}$$
$$+ \frac{1-\delta_{AB}}{\lambda_A \lambda_B}[-P + 4\lambda_A^2 \lambda_B^2 W_2] + 4\lambda_A \lambda_B [W_{11} + (2I_1 - \lambda_A^2 - \lambda_B^2) W_{12} + (I_1 - \lambda_A^2)(I_1 - \lambda_B^2) W_{22}]$$

$$K_{ABBA} = K_{BAAB} = 2[W_1 + (I_1 - \lambda_A^2 - \lambda_B^2) W_2], \quad (A \neq B)$$

$$K_{ABAB} = K_{BABA} = \frac{1}{\lambda_A \lambda_B}(P - 2\lambda_A^2 \lambda_B^2 W_2), \quad (A \neq B).$$